

Wiener's problem for positive definite functions

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ABSTRACT. We study the sharp constant $W_n(D)$ in Wiener's inequality for positive definite functions

$$\int_{\mathbb{T}^n} |f|^2 dx \leq W_n(D) |D|^{-1} \int_D |f|^2 dx, \quad D \subset \mathbb{T}^n.$$

N. Wiener proved that $W_1([-\delta, \delta]) < \infty$, $\delta \in (0, 1/2)$. E. Hlawka showed that $W_n(D) \leq 2^n$, where D is an origin-symmetric convex body.

We sharpen Hlawka's estimates for D being the ball B^n and the cube I^n . In particular, we prove that $W_n(B^n) \leq 2^{(0.401\dots + o(1))n}$. We also obtain a lower bound of $W_n(D)$. Moreover, for a cube $D = \frac{1}{q}I^n$ with $q = 3, 4, \dots$, we obtain that $W_n(D) = 2^n$. Our proofs are based on the interrelation between Wiener's problem and the problems of Turán and Delsarte.

1. Introduction

Let $n \in \mathbb{N}$ and $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$. The Fourier series of a complex-valued function $f \in L^1(\mathbb{T}^n)$ is given by

$$f(x) = \sum_{\nu \in \mathbb{Z}^n} \widehat{f}_\nu e(\nu x), \quad e(t) = e^{2\pi i t},$$

where

$$\widehat{f}_\nu = \int_{\mathbb{T}^n} f(x) e(-\nu x) dx, \quad \nu \in \mathbb{Z}^n,$$

are the Fourier coefficients of f . The support of a function f , written $\text{supp } f$, is the closure of the subset of \mathbb{T}^n where f is non-zero. Let the unit ball and the unit cube be given by $B^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ and $I^n = [-1, 1]^n$, respectively. Let also $B_r := B_r^n := rB^n$ for $r > 0$. By $|D|$ we denote the volume of $D \subset \mathbb{R}^n$. In what follows, we assume that D is an origin-symmetric convex body.

Key words and phrases. positive definite function, Wiener's problem, Hlawka's inequality, sharp constant, linear programming bound problem.

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Wiener's inequality for positive definite functions in $L^2(\mathbb{T}^n)$ is given by

$$(1) \quad \int_{\mathbb{T}^n} |f(x)|^2 dx \leq C_n(D) \int_D |f(x)|^2 dx, \quad f \in L_+^1(\mathbb{T}^n),$$

where

$$L_+^1(\mathbb{T}^n) := \{f \in L^1(\mathbb{T}^n) : \widehat{f}_\nu \geq 0 \text{ for any } \nu \in \mathbb{Z}^n\},$$

and $D \subset \mathbb{T}^n$. Here $C_n(D)$ is a positive constant depending only on n and D . Note that $L_+^1(\mathbb{T}^n) \not\subset L^2(\mathbb{T}^n)$, take, for example,

$$f(x) = \sum_{k=1}^{\infty} k^{-1/2} \cos(2\pi k x_1),$$

see [Zy, Ch. V, (1.8)].

N. Wiener (unpublished result, see e.g. [Sh]) proved in the early 1950's that $C_1([-\delta, \delta]) < \infty$ for $\delta \in (0, 1/2)$.

For $n = 1$, H. Shapiro [Sh] showed that, for any $\delta \in (0, 1/2)$,

$$\int_{\mathbb{T}} |f|^2 dx \leq \delta^{-1} \int_{-\delta}^{\delta} |f|^2 dx, \quad f \in L_+^1(\mathbb{T}).$$

The latter was generalized by E. Hlawka [Hl] for the multivariate case as follows:

$$(2) \quad \int_{\mathbb{T}^n} |f|^2 dx \leq |\tfrac{1}{2}D|^{-1} \int_D |f|^2 dx, \quad f \in L_+^1(\mathbb{T}^n),$$

where $D \subset \mathbb{T}^n$ is an origin-symmetric convex body.

The goal of this paper is to study the sharp constant

$$W_n(D) := \sup_{f \in L_+^1(\mathbb{T}^n) \setminus \{0\}} \frac{\int_{\mathbb{T}^n} |f|^2 dx}{|D|^{-1} \int_D |f|^2 dx}.$$

Note that Hlawka's result implies that

$$(3) \quad W_n(D) \leq 2^n, \quad n \in \mathbb{N}.$$

Moreover, taking $f = 1$ we get a trivial estimate from below

$$1 \leq W_n(D)$$

for any $D \subset \mathbb{T}^n$.

Note that $f \in L_+^1(\mathbb{T}^n)$ if and only if f is positive definite [Ed, 9.2.4]. Recall that an integrable function f is positive definite [Ed, Chap. 9] if

$$(4) \quad \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} f(x-y) u(x) \overline{u(y)} dx dy \geq 0$$

for any $u \in C(\mathbb{T}^n)$. It is sufficient to verify (4) only for the case of u being trigonometric polynomials.

For a continuous function $f \in C(\mathbb{T}^n)$, condition (4) is equivalent to the fact that f is positive definite in the classical sense, that is, for every finite

sequence $\{x_i\}_{i=1}^N$ in \mathbb{T}^n and every choice of complex numbers $\{c_i\}_{i=1}^N$, we have

$$\sum_{i,j=1}^N c_i \overline{c_j} f(x_i - x_j) \geq 0,$$

see [Ru, Chap. 1].

Note that if a function $f \in L_+^1(\mathbb{T}^n)$ is bounded in some neighborhood of the origin and therefore the series $\sum_{\nu} \widehat{f}_{\nu}$ converges, then Bochner's theorem [Ed, 9.2.8] implies that f can be viewed as a continuous positive definite function. In general, this is not the case. However, the following result is true (see Section 4 for its proof).

PROPOSITION 1. *We have*

$$(5) \quad W_n(D) = W_n^+(D) := \sup_{f \in F_+ \setminus \{0\}} \frac{\int_{\mathbb{T}^n} |f|^2 dx}{|D|^{-1} \int_D |f|^2 dx},$$

where F_+ is a set of continuous positive definite functions on \mathbb{T}^n .

In this paper, we continue investigating multivariate inequality (2) and prove new bounds for $W_n(D)$. Moreover, we connect this problem to Turán's problem (see Section 2) and Delsarte's problem also known as the linear programming bound problem (see Section 3).

The main results of the paper are new bounds of $W_n(D)$ in the case when D is a ball or a cube.

THEOREM 2. *For $\delta \in (0, 1/2)$, we have*

$$W_n(\delta B^n) \leq 2^{(0.401\dots + o(1))n}.$$

THEOREM 3. *Let $\delta \in (0, 1/2)$. Then*

$$W_n(\delta I^n) \leq 2^n (1 - \theta(\delta))^n,$$

where

- (i) $\theta(\delta) \in C(0, 1/2)$, and, moreover, $\theta(\delta) = O(\delta^2)$ as $\delta \rightarrow 0$,
- (ii) $\theta(\delta) = 0$ for $\delta^{-1} \in \mathbb{N}$,
- (iii) $0 < \theta(\delta) < 1$ for $\delta^{-1} \notin \mathbb{N}$.

Note that there is a specific expression for $\theta(\delta)$, which is

$$\theta(\delta) = 1 - \delta a_{\mathbb{T}}^{-1}([-\delta, \delta]),$$

where $a_{\mathbb{T}}([-\delta, \delta])$ is the solution of Turán's problem, see Section 2.

Our next result provides an estimate of $W_n(D)$ from below.

THEOREM 4. *Let $D \subset \delta I^n$ with $\delta \in (0, 1/q)$ for some $q = 2, 3, \dots$. Then*

$$|D|q^n \leq W_n(D).$$

For the cube $D = \delta I^n$, this result gives

$$(6) \quad 2^n (\delta q)^n \leq W_n(\delta I^n), \quad \delta \in (0, 1/q).$$

Letting $\delta \rightarrow \frac{1}{q}-$, this and Hlawka's inequality (3) give

COROLLARY 5 (Wiener's constant for the cube). *For $q = 3, 4, \dots$, we have*

$$W_n(\frac{1}{q}I^n) = 2^n.$$

It is worth mentioning that setting¹ $q = [\delta^{-1}]$ for $\delta^{-1} \notin \mathbb{N}$ and $q = [\delta^{-1}] - 1$ for $\delta^{-1} \in \mathbb{N}$ in (6), we obtain

$$2^n(1 - \delta)^n \leq W_n(\delta I^n), \quad \delta \in (0, 1/2).$$

In particular, this implies that the limit

$$\lim_{\delta \rightarrow 0+} W_n(\delta I^n) = 2^n$$

exists. It is not known if this limit exists for other D .

2. Wiener's problem vs. Turán's problem

The periodic Turán problem in $L^1(\mathbb{T}^n)$ for positive definite functions consists of finding [Go2]

$$a_{\mathbb{T}^n}(D) = \sup \widehat{g}_0,$$

where supremum is taken over all functions $g \in L^1(\mathbb{T}^n)$ such that

$$(7) \quad \widehat{g}_\nu \geq 0, \quad \text{supp } g \subset D, \quad g(0) = 1.$$

Similarly we introduce the non-periodic Turán problem in $L^1(\mathbb{R}^n)$ for positive definite functions with compact support:

$$a_{\mathbb{R}^n}(D) = \sup \widehat{u}(0),$$

where

$$h \in L^1(\mathbb{R}^n), \quad \widehat{h} \geq 0, \quad \text{supp } h \subset D, \quad h(0) = 1.$$

Here D is any subset of \mathbb{R}^n and

$$\widehat{h}(\xi) = \int_{\mathbb{R}^n} h(x) e(-\xi x) dx, \quad \xi \in \mathbb{R}^n,$$

is the Fourier transform of h .

Note that Turán's problem is closely related to Boas–Kac–Krein representation theorem on convolution roots of positive definite functions with compact support, which in turn has applications to geostatistical simulation, crystallography, optics, and phase retrieval (see [EGR]) as well as Fuglede's conjecture (see [KR1]). Very recent results on the topic can be found in [KR2].

The periodic and spatial problems are connected as follows [Go2]:

$$a_{\mathbb{R}^n}(D) \leq \lambda^{-n} a_{\mathbb{T}^n}(\lambda D) \leq a_{\mathbb{R}^n}(D)(1 + O(\lambda^2)), \quad \lambda \in (0, 1].$$

¹As usual, by $[a]$ we denote the integer part of $a \in \mathbb{R}$.

The periodic Turán problem was studied in one dimension in [GM, IGR, IR] and completely solved in [Iv]. Since the solution has a complicated form, we only highlight the following two facts. If $q = 2, 3, \dots$, then

$$(8) \quad a_{\mathbb{T}}([-1/q, 1/q]) = 1/q$$

(see also [St]).

If $\delta \in (0, 1/2)$, then

$$(9) \quad \begin{aligned} a_{\mathbb{T}}([- \delta, \delta]) &> \delta, & \delta^{-1} &\notin \mathbb{N}, \\ a_{\mathbb{T}}([- \delta, \delta]) &= \delta(1 + O(\delta^2)), & \delta &\rightarrow 0. \end{aligned}$$

Now we are in a position to sharpen the known bounds of $W_n(D)$ from above, cf. (3).

THEOREM 6. *Let $D \subset \mathbb{T}^n$. We have*

$$|D|^{-1}W_n(D) \leq (a_{\mathbb{T}^n}(D))^{-1} \leq (a_{\mathbb{R}^n}(D))^{-1} \leq |\tfrac{1}{2}D|^{-1}.$$

PROOF. First, we show that

$$(10) \quad |D|^{-1}W_n(D) \leq (a_{\mathbb{T}^n}(D))^{-1}.$$

Let g be an admissible function for the periodic Turán problem, i.e., g satisfies condition (7). Then since g is positive definite, we have $\text{supp } g \subset D$ and $g(x) \leq g(0) = 1$, $x \in \mathbb{T}^n$. Hence for any $f \in L_+^1(\mathbb{T}^n)$ we get

$$\int_D |f|^2 dx \geq \int_{\mathbb{T}^n} |f|^2 g dx.$$

Note that $fg \in L^1(\mathbb{T}^n) \cap L^2(\mathbb{T}^n)$. Since both f and g have nonnegative Fourier coefficients, we obtain that

$$(\widehat{fg})_\nu = \sum_\mu \widehat{f}_{\nu-\mu} \widehat{g}_\mu \geq \widehat{g}_0 \widehat{f}_\nu$$

and

$$(11) \quad \begin{aligned} \int_{\mathbb{T}^n} |f|^2 g dx &= \int_{\mathbb{T}^n} \overline{f(x)} \sum_\nu (\widehat{fg})_\nu e(\nu x) dx = \sum_\nu (\widehat{fg})_\nu \int_{\mathbb{T}^n} \overline{f(x)} e(\nu x) dx \\ &= \sum_\nu (\widehat{fg})_\nu \widehat{\overline{f}}_\nu \geq \widehat{g}_0 \sum_\nu |\widehat{f}_\nu|^2 = \widehat{g}_0 \int_{\mathbb{T}^n} |f|^2 dx. \end{aligned}$$

Thus,

$$(12) \quad \int_{\mathbb{T}^n} |f|^2 dx \leq (\widehat{g}_0)^{-1} \int_D |f|^2 dx,$$

which gives $|D|^{-1}W_n(D) \leq (\widehat{g}_0)^{-1}$. Minimizing $(\widehat{g}_0)^{-1}$, or equivalently maximizing \widehat{g}_0 , we arrive at (10).

Secondly, we prove that $a_{\mathbb{R}^n}(D) \leq a_{\mathbb{T}^n}(D)$. Let h be any admissible function in the spatial Turán problem. Consider the periodic function [SW, Chap. 7]

$$g(x) = \sum_{\nu \in \mathbb{Z}^n} h(x + \nu).$$

Then $\hat{g}_\nu = \hat{h}(\nu) \geq 0$, $\text{supp } g \subset D$, and $g(0) = \sum_{\nu \in \mathbb{Z}^n} h(\nu) = h(0) = 1$. Therefore, g is an admissible function in the periodic Turán problem and, moreover, $\hat{h}(0) = \hat{g}_0 \leq a_{\mathbb{T}^n}(D)$, which implies $a_{\mathbb{R}^n}(D) \leq a_{\mathbb{T}^n}(D)$.

Third, to show that $(a_{\mathbb{R}^n}(D))^{-1} \leq |K|^{-1}$, where $K = \frac{1}{2}D$, we set

$$(13) \quad h_* = b^{-1} \chi_K * \chi_K,$$

where $b = (\chi_K * \chi_K)(0) = |K|$. Then $\text{supp } h_* \subset D$ and $\hat{h}_* = b^{-1} \hat{\chi}_K^2 \geq 0$. Moreover,

$$(14) \quad a_{\mathbb{R}^n}(D) \geq \hat{h}_*(0) = |K|^{-1} \hat{\chi}_K^2(0) = |K|^{-1} \left(\int_K dx \right)^2 = \left| \frac{1}{2}D \right|,$$

which completes the proof. \square

There is a conjecture [BK] that

$$a_{\mathbb{R}^n}(D) = \left| \frac{1}{2}D \right|$$

and h_* is an extremal function. This conjecture was proved only for the ball and Voronoi polytopes of lattices. In the case of the ball this was first done by Siegel [Si] and later in [Go2, KR1, BK]). For the case of the Voronoi polytopes see [AB1, AB2]. It is worth mentioning that these results are also known for any rotation and scaling of D . This follows from

REMARK 1. Let f be an admissible function in Turán problem, $\rho \in SO(n)$, $\lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$, $\lambda_i > 0$, and $g(x) = f(\lambda^{-1} \rho^{-1} x)$, $x \in \mathbb{R}^n$. Then $\text{supp } g \subset \rho \lambda D$ and $\hat{g}(\xi) = \det \lambda \cdot \hat{f}(\lambda \rho^{-1} \xi) \geq 0$, where $\xi \in \mathbb{R}^n$ and $\det \lambda = \lambda_1 \dots \lambda_n$.

We have that g is a positive definite function such that $\hat{g}(0) = \det \lambda \cdot \hat{f}(0)$ and $g(0) = f(0) = 1$. Thus, we get

$$a_{\mathbb{R}^n}(\lambda \rho D) = \det \lambda \cdot a_{\mathbb{R}^n}(D).$$

Theorem 6 does not provide an improvement of Hlawka's inequality $W_n(D) \leq 2^n$ in the case of the ball. In order to sharpen this bound we will now consider a wider class of admissible functions h .

3. Wiener's problem vs. Delsarte's problem

By the Delsarte problem in $L^1(\mathbb{R}^n)$ for positive definite functions we mean the following question [Go1, CE]:

$$(15) \quad A_{\mathbb{R}^n}(D) = \inf h(0),$$

where infimum is taken over all functions $h \in L^1(\mathbb{R}^n)$ such that

$$\hat{h} \geq 0, \quad h|_{\mathbb{R}^n \setminus D} \leq 0, \quad \hat{h}(0) = 1.$$

Noting that $\widehat{h}_\lambda(\xi) = \lambda^{-n} \widehat{h}(\lambda^{-1}\xi)$, where $h_\lambda(x) = h(\lambda x)$, we have

$$(16) \quad A_{\mathbb{R}^n}(\lambda D) = \lambda^{-n} A_{\mathbb{R}^n}(D), \quad \lambda > 0.$$

The importance of Delsarte's problem can be illustrated by the following remarks. Let $D = 2B^n$ and Δ_n be the center density of sphere packings in \mathbb{R}^n (see the details in [Go1, CE, Co]).

The best known density bounds are

$$2^{-(1+o(1))n} \leq \Delta_n \leq 2^{-(0.5990\dots+o(1))n} =: C_{KL}$$

as $n \rightarrow \infty$. For the best known lower estimate see [Ve]. The upper bound was found by G. Kabatiansky and V. Levenshtein in [KL].

In [Le], V. Levenshtein proved that

$$\Delta_n \leq \frac{(q_{n/2}/4)^n}{\Gamma^2(n/2 + 1)} = 2^{-(0.5573\dots+o(1))n} =: C_L,$$

where $q_{n/2}$ is the first positive zero of the Bessel function $J_{n/2}(t)$.

Later, D. Gorbachev [Go1] and H. Cohn and N. Elkies [Co, CE] proved the linear programming bounds

$$\Delta_n \leq |B^n| A_{\mathbb{R}^n}(2B^n) =: C_A.$$

In particular, this yields that

$$C_A \leq C_L.$$

Moreover, for admissible functions h for the Delsarte problem, that is satisfying (15), and such that $\text{supp } \widehat{h} \subset \rho_n B^n$, where $\rho_n = q_{n/2}/(2\pi)$, we get $C_A = C_L$.

Recently, H. Cohn and Y. Zhao [CZ] proved that

$$(17) \quad C_A \leq C_{KL}.$$

In 2016, C_A was calculated when $n = 8$ and $n = 24$ (see [Vi] and [CKMRV] respectively), and moreover, $\Delta_n = C_A$ for such n .

Our main result in this section is a new bound of Wiener's constant.

THEOREM 7. *For $D \subset \mathbb{T}^n$ we have*

$$|D|^{-1} W_n(D) \leq A_{\mathbb{R}^n}(D) \leq (a_{\mathbb{R}^n}(D))^{-1}.$$

PROOF. Let us first show that $A_{\mathbb{R}^n}(D) \leq (a_{\mathbb{R}^n}(D))^{-1}$. We have

$$(A_{\mathbb{R}^n}(D))^{-1} = \sup \widehat{h}(0),$$

where $\widehat{h} \geq 0$, $h|_{\mathbb{R}^n \setminus D} \leq 0$, and $h(0) = 1$. This problem differs from the Turán problem only by a less restrictive condition $h|_{\mathbb{R}^n \setminus D} \leq 0$ in place of $h|_{\mathbb{R}^n \setminus D} = 0$. Hence, $(A_{\mathbb{R}^n}(D))^{-1} \geq a_{\mathbb{R}^n}(D)$.

It is enough to show that

$$(18) \quad |D|^{-1} W_n(D) \leq A_{\mathbb{R}^n}(D).$$

Let h be an admissible function for the Delsarte problem. Consider

$$g(x) = \sum_{\nu \in \mathbb{Z}^n} h(x + \nu), \quad x \in \mathbb{T}^n.$$

This is a positive definite function on \mathbb{T}^n and therefore $g(x) \leq g(0)$ for any $x \in \mathbb{T}^n$. Since $D \subset \mathbb{T}^n$, we have $h(x + \nu) \leq 0$ for $x \in \mathbb{T}^n \setminus D$ and $h(\nu) \leq 0$ for $\nu \neq 0$, $\nu \in \mathbb{Z}^n$. Thus we have that $g|_{\mathbb{T}^n \setminus D} \leq 0$ and $g(0) \leq h(0)$.

Then, following the proof of Theorem 6, for any positive definite function $f \in L^1(\mathbb{T}^n)$ we obtain

$$\int_D |f|^2 dx \geq (g(0))^{-1} \int_D |f|^2 g dx \geq (g(0))^{-1} \int_{\mathbb{T}^n} |f|^2 g dx.$$

Hence, using (11), for positive definitive functions f and g , we have

$$\int_{\mathbb{T}^n} |f|^2 g dx \geq \hat{g}_0 \int_{\mathbb{T}^n} |f|^2 dx$$

and $\hat{g}_0 = \hat{h}(0) = 1$. Then

$$\int_D |f|^2 dx \geq (g(0))^{-1} \int_{\mathbb{T}^n} |f|^2 dx \geq (h(0))^{-1} \int_{\mathbb{T}^n} |f|^2 dx,$$

which implies $|D|^{-1} W_n(D) \leq h(0)$. Taking infimum over all h , we conclude the proof of (18). \square

4. Proofs of main results

PROOF OF PROPOSITION 1. First, since $F_+ \subset L_+^1$, we always have $W_n^+(D) \leq W_n(D)$.

To prove $W_n(D) \leq W_n^+(D)$, let $f \in L_+^1(\mathbb{T}^n) \cap L^2(D)$, $f \neq 0$. We will show that to study the supremum in (5) it is enough to consider continuous functions f satisfying $|f(0)| \leq \|f\|_{L^2(D)}$.

Let us consider a non-negative continuous positive definite radial function $\varphi(x)$, $x \in \mathbb{R}^n$, such that $\text{supp } \varphi \subset B^n$ and $\widehat{\varphi}(0) = 1$. For example, one can put

$$\varphi(x) = h_*(x) / \widehat{h}_*(0),$$

where h_* is given by (13) with $K = \frac{1}{2}B^n$. Note that in this case, h_* is radial since χ_K is radial.

For every small $\varepsilon > 0$ such that $B_\varepsilon \subset D$ let us define

$$\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(\varepsilon^{-1}x).$$

This function satisfies the following conditions

$$(19) \quad \text{supp } \varphi_\varepsilon \subset B_\varepsilon, \quad \widehat{\varphi}_\varepsilon(\xi) = \widehat{\varphi}(\varepsilon\xi), \quad \xi \in \mathbb{R}^n, \quad \widehat{\varphi}_\varepsilon(0) = 1.$$

Now we set

$$(20) \quad \psi_\varepsilon(x) = \sum_{\nu \in \mathbb{Z}^n} \varphi_\varepsilon(x + \nu), \quad x \in \mathbb{T}^n.$$

Then, by (19), this function satisfies the following conditions

- (a) $\text{supp } \psi_\varepsilon \subset B_\varepsilon$,
- (b) $(\widehat{\psi_\varepsilon})_\nu = \widehat{\varphi_\varepsilon}(\nu) = \widehat{\varphi}(\varepsilon\nu), \quad \nu \in \mathbb{Z}^n$,
- (c) $(\widehat{\psi_\varepsilon})_0 = 1$.

Thus, $\psi_\varepsilon \in L_+^1(\mathbb{T}^n) \cap C(\mathbb{T}^n)$.

Define

$$f_\varepsilon(x) = b(f * \psi_\varepsilon)(x), \quad x \in \mathbb{T}^n,$$

where

$$b^{-1} = \|\psi_\varepsilon\|_{L^2(\mathbb{T}^n)} = \|\psi_\varepsilon\|_{L^2(B_\varepsilon)}.$$

Then we have that $(\widehat{f_\varepsilon})_\nu = b\widehat{f}_\nu(\widehat{\psi_\varepsilon})_\nu$. Moreover, $f_\varepsilon \in F_+$, which follows from Young's inequality

$$\|f_\varepsilon\|_{C(\mathbb{T}^n)} \leq b\|f\|_{L^2(\mathbb{T}^n)}\|\psi_\varepsilon\|_{L^2(\mathbb{T}^n)} = \|f\|_{L^2(\mathbb{T}^n)}.$$

Moreover,

$$|f_\varepsilon(0)| = b \left| \int_{B_\varepsilon} f \psi_\varepsilon dx \right| \leq b\|f\|_{L^2(B_\varepsilon)}\|\psi_\varepsilon\|_{L^2(B_\varepsilon)} \leq \|f\|_{L^2(D)}.$$

The function ψ_ε is non-negative such that its mean value is equal to 1. Then by Hölder's inequality we get that

$$\left| \int_{B_\varepsilon} f(x-y) \psi_\varepsilon(y) dy \right|^2 \leq \int_{B_\varepsilon} |f(x-y)|^2 \psi_\varepsilon(y) dy$$

for any fixed x . Then it follows that

$$\|f_\varepsilon\|_{L^2(D)}^2 = b^2 \int_D \left| \int_{B_\varepsilon} f(x-y) \psi_\varepsilon(y) dy \right|^2 dx \leq b^2 \int_{B_\varepsilon} \psi_\varepsilon(y) g(y) dy,$$

where

$$g(y) = \int_D |f(x-y)|^2 dx = (|f|^2 * \chi_D)(-y).$$

The function g is continuous in a neighborhood of the origin, since $|f|^2 \in L^1(\mathbb{T}^n)$. Therefore, for any $\varepsilon' > 0$ there exists $\varepsilon > 0$ such that

$$|g(y)| \leq (1 + \varepsilon')g(0) = (1 + \varepsilon')\|f\|_{L^2(D)}^2, \quad |y| < \varepsilon,$$

which gives

$$\|f_\varepsilon\|_{L^2(D)}^2 \leq b^2(1 + \varepsilon')\|f\|_{L^2(D)}^2.$$

By Parseval's identity,

$$\|f_\varepsilon\|_{L^2(\mathbb{T}^n)}^2 = b^2 \sum_{\nu} (\widehat{f}_\nu)^2 (\widehat{\varphi}(\varepsilon\nu))^2.$$

Moreover, by Hlawka's inequality (2) (or by Theorem 6), we have

$$\sum_{\nu} (\widehat{f}_\nu)^2 = \|f\|_{L^2(\mathbb{T}^n)}^2 \leq |\tfrac{1}{2}D|^{-1} \|f\|_{L^2(D)}^2.$$

Using $\widehat{\varphi}^2(\varepsilon\nu) \leq \widehat{\varphi}^2(0) = 1$, we have that

$$\sum_{\nu} (\widehat{f}_\nu)^2 (\widehat{\varphi}(\varepsilon\nu))^2 \rightarrow \|f\|_{L^2(\mathbb{T}^n)}^2$$

uniformly as $\varepsilon \rightarrow 0$. Hence for any small $\varepsilon'' > 0$ we can find $\varepsilon > 0$ such that

$$\|f_\varepsilon\|_{L^2(\mathbb{T}^n)}^2 \geq b^2(1 - \varepsilon'')\|f\|_{L^2(\mathbb{T}^n)}^2.$$

Finally,

$$\frac{\|f\|_{L^2(\mathbb{T}^n)}^2}{|D|^{-1}\|f\|_{L^2(D)}^2} \leq \frac{b^{-2}(1 - \varepsilon'')^{-1}\|f_\varepsilon\|_{L^2(\mathbb{T}^n)}^2}{|D|^{-1}b^{-2}(1 + \varepsilon')^{-1}\|f_\varepsilon\|_{L^2(D)}^2} \leq \frac{1 + \varepsilon'}{1 - \varepsilon''} W_n^+(D).$$

Letting $\varepsilon', \varepsilon'' \rightarrow 0$ gives the required result. \square

PROOF OF THEOREM 2. Combining property (16), Cohn-Zhao's estimate (17), and Theorem 7, we arrive at

$$\begin{aligned} W_n(\delta B^n) &\leq |\delta B^n| A_{\mathbb{R}^n}(\delta B^n) = 2^n |B^n| A_{\mathbb{R}^n}(2B^n) \\ &= 2^n C_A \leq 2^n C_{KL}, \end{aligned}$$

where $2^n C_{KL} = 2^{(0.401\dots + o(1))n}$. \square

To prove Theorem 3, we need the following

LEMMA 1. *Let $\mathbb{T}^n = \mathbb{T}^{n_1} \times \mathbb{T}^{n_2}$, $D_i \subset \mathbb{T}^{n_i}$, $i = 1, 2$. Then*

$$a_{\mathbb{T}^n}(D_1 \times D_2) = a_{\mathbb{T}^{n_1}}(D_1) a_{\mathbb{T}^{n_2}}(D_2).$$

We note that for the spatial Turán problem a similar fact is known, see [AB2]. To make the paper self-contained we prove Lemma 1 using the same idea as in [AB2].

PROOF OF LEMMA 1. Let $D = D_1 \times D_2$, $x = (x_1, x_2) \in \mathbb{T}^n$, where $x_i \in \mathbb{T}^{n_i}$, $i = 1, 2$.

We start by assuming $f_i(x_i)$ to be admissible functions in the problem $a_{\mathbb{T}^{n_i}}(D_i)$. Then the function $f(x) = f_1(x_1)f_2(x_2)$ is also an admissible function in the problem $a_{\mathbb{T}^n}(D)$, since $f(0) = 1$, $f(x) = 0$ for $x_i \notin D_i$, and $\hat{f}_\nu = (\hat{f}_1)_{\nu_1}(\hat{f}_2)_{\nu_2} \geq 0$, $\nu = (\nu_1, \nu_2) \in \mathbb{Z}^n$.

Therefore, we have

$$a_{\mathbb{T}^n}(D) \geq \hat{f}_0 = (\hat{f}_1)_0(\hat{f}_2)_0,$$

which gives that

$$a_{\mathbb{T}^n}(D) \geq a_{\mathbb{T}^{n_1}}(D_1) a_{\mathbb{T}^{n_2}}(D_2).$$

On the other hand, if $f(x)$ is an admissible function in the problem $a_{\mathbb{T}^n}(D)$, then we define $f_1(x_1) = f(x_1, 0)$ and

$$f_2(x_2) = b^{-1} \int_{\mathbb{T}^{n_1}} f(x_1, x_2) dx_1, \quad b = \int_{\mathbb{T}^{n_1}} f(x_1, 0) dx_1.$$

Let us show that the functions $f_i(x_i)$ are admissible in the problems $a_{\mathbb{T}^{n_i}}(D_i)$.

First, we have that $\text{supp } f_i \subset D_i$, $f_1(0) = f(0) = 1$, $f_2(0) = 1$. The function f_1 is positive definite since f is positive definite and

$$f_1(x_1) = f(x_1, 0) = \sum_{\nu_1 \in \mathbb{Z}^{n_1}} (\hat{f}_1)_{\nu_1} e(\nu_1 x_1),$$

where

$$0 \leq (\widehat{f_1})_{\nu_1} = \sum_{\nu_2 \in \mathbb{Z}^{n_2}} \widehat{f_{\nu_1, \nu_2}} \leq \sum_{\nu \in \mathbb{Z}^n} \widehat{f_\nu} = 1.$$

The function f_2 is positive definite since $b = (\widehat{f_1})_0 > 0$ and $(\widehat{f_2})_{\nu_2} = b^{-1} \widehat{f_{0, \nu_2}} \geq 0$.

We have

$$\widehat{f_0} = b(\widehat{f_2})_0 = (\widehat{f_1})_0(\widehat{f_2})_0 \leq a_{\mathbb{T}^{n_1}}(D_1)a_{\mathbb{T}^{n_2}}(D_2).$$

Thus

$$a_{\mathbb{T}^n}(D) \leq a_{\mathbb{T}^{n_1}}(D_1)a_{\mathbb{T}^{n_2}}(D_2).$$

□

PROOF OF THEOREM 3. To show the estimate of $W_n(D)$ from above we use Lemma 1 to get

$$(21) \quad a_{\mathbb{T}^n}([-\delta, \delta]^n) = (a_{\mathbb{T}}([-\delta, \delta]))^n, \quad \delta \in (0, 1/2).$$

Let $\delta \in (0, 1/2)$ and $\theta(\delta) = 1 - \delta a_{\mathbb{T}}^{-1}([-\delta, \delta])$. Using Theorem 6, and property (21), we get

$$\begin{aligned} W_n(D) &\leq |D|(a_{\mathbb{T}}([-\delta, \delta]))^{-n} = 2^n(\delta a_{\mathbb{T}}^{-1}([-\delta, \delta]))^n \\ &= 2^n(1 - \theta(\delta))^n, \end{aligned}$$

completing the proof. □

PROOF OF THEOREM 4. Let $D \subset \delta I^n$, $\delta \in (0, 1/q)$, $q = 2, 3, \dots$, $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$, and $\Gamma = \{\nu/q : \nu \in \mathbb{Z}_q^n\} \subset \mathbb{T}^n$. We have $|\Gamma| = q^n$ and

$$s_\nu := |\Gamma|^{-1} \sum_{\gamma \in \Gamma} e(\nu\gamma) = \prod_{i=1}^n q^{-1} \sum_{k=0}^{q-1} e\left(\frac{\nu_i k}{q}\right) \in \{0, 1\},$$

where $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{Z}^n$.

Let $0 < \varepsilon < \min\{\delta, 1/q - \delta\}$. Taking into account that coordinates of lattice points Γ are multiple of $1/q$, and $\varepsilon < 1/(2q)$, we have that

$$(B_\varepsilon + \Gamma) \cap D = B_\varepsilon.$$

Moreover, for any $\gamma, \gamma' \in \Gamma$ the sets $B_\varepsilon + \gamma$ and $B_\varepsilon + \gamma'$ are disjoint.

Now we will use the positive definite function ψ_ε given by (20) and satisfying $\text{supp } \psi_\varepsilon \subset B_\varepsilon$. We define

$$(22) \quad f(x) = |\Gamma|^{-1} \sum_{\gamma \in \Gamma} \psi_\varepsilon(x - \gamma).$$

Then f is a periodic function supported on $B_\varepsilon + \Gamma$ and such that

$$\widehat{f}_\nu = (\widehat{\psi_\varepsilon})_\nu s_{-\nu} \geq 0, \quad \nu \in \mathbb{Z}^n.$$

This gives the positive definiteness of f .

Now we note that supports of the functions $\psi_\varepsilon(x - \gamma)$, which are equal to $B_\varepsilon + \gamma$, are disjoint. Using this, we obtain

$$(23) \quad f^2(x) = |\Gamma|^{-2} \sum_{\gamma \in \Gamma} \psi_\varepsilon^2(x - \gamma), \quad x \in \mathbb{T}^n.$$

Integrating this and taking into account that

$$\int_{\mathbb{T}^n} \psi_\varepsilon^2(x - \gamma) dx = \int_{\mathbb{T}^n} \psi_\varepsilon^2(x) dx = \|\psi_\varepsilon\|_{L^2(B_\varepsilon)}^2,$$

we get

$$(24) \quad \int_{\mathbb{T}^n} f^2 dx = |\Gamma|^{-2} \sum_{\gamma \in \Gamma} \int_{\mathbb{T}^n} \psi_\varepsilon^2(x - \gamma) dx = |\Gamma|^{-1} \|\psi_\varepsilon\|_{L^2(B_\varepsilon)}^2.$$

In light of $(B_\varepsilon + \Gamma) \cap D = B_\varepsilon$ we have

$$(25) \quad \int_D f^2 dx = |\Gamma|^{-2} \int_{B_\varepsilon} \psi_\varepsilon^2 dx = |\Gamma|^{-2} \|\psi_\varepsilon\|_{L^2(B_\varepsilon)}^2.$$

Thus,

$$(26) \quad W_n(D) \geq \frac{\int_{\mathbb{T}^n} f^2 dx}{|D|^{-1} \int_D f^2 dx} = |D| |\Gamma| = |D| q^n.$$

□

5. Wiener's inequality in \mathbb{R}^n

Let $f: \mathbb{R}^n \rightarrow \mathbb{C}$ be a positive definite function in the following sense: f is integrable and such that $\hat{f} \geq 0$. Then a formal analogue of Wiener's inequality (1) given by

$$\int_{\mathbb{R}^n} |f|^2 dx \leq C_n(D) \int_D |f|^2 dx,$$

for any origin-symmetric convex body $D \subset \mathbb{R}^n$, does not hold in general. It is enough to consider the non-negative function (13)

$$f = |B_r|^{-1} \chi_{B_r} * \chi_{B_r}$$

for sufficiently large r . Indeed, for $f(x) \leq f(0) = 1$, $\text{supp } f \subset B_{2r}$, we have

$$\int_{\mathbb{R}^n} |f|^2 dx = \int_{B_{2r}} |f|^2 dx \geq |B_{2r}|^{-1} \left(\int_{B_{2r}} f dx \right)^2.$$

Moreover, similarly to (14),

$$\int_{B_{2r}} f dx = \hat{f}(0) = |B_r|^{-1} \hat{\chi}_{B_r}^2(0) = |B_r|.$$

Then

$$\int_{\mathbb{R}^n} |f|^2 dx \geq 2^{-n} |B_r| \rightarrow \infty, \quad r \rightarrow \infty.$$

THEOREM 8. *Let $f \in L^1(\mathbb{R}^n)$ be a positive definite function. Then for $D \subset \mathbb{T}^n$ we have*

$$(27) \quad \int_{\mathbb{R}^n} |f|^2 dx \leq (a_{\mathbb{R}^n}(D))^{-1} \int_{D+\mathbb{Z}^n} |f|^2 dx.$$

Theorems 6 and 8 immediately imply

COROLLARY 9 (Hlawka's inequality in \mathbb{R}^n). *Under conditions of Theorem 8, we have*

$$\int_{\mathbb{R}^n} |f|^2 dx \leq |\tfrac{1}{2}D|^{-1} \int_{D+\mathbb{Z}^n} |f|^2 dx.$$

This result is new even in the one-dimensional case; a weaker estimate was proved in [KOT, Th. 3.3].

REMARK 2. Theorem 8 and Corollary 9 hold for positive definite functions defined in the usual way: $f \in C(\mathbb{R}^n)$ and

$$(28) \quad f(x) = f(0) \int_{\mathbb{R}^n} e(x\xi) d\mu(\xi), \quad f(0) \geq 0,$$

where μ is a finite positive measure on R , see [Ru, Chap. 1].

PROOF OF THEOREM 8. Let $f \in L^1(\mathbb{R}^n)$ and $\hat{f} \geq 0$. Suppose that $h \in L^1(\mathbb{R}^n)$ is an admissible function in the spatial Turán problem, that is, $\hat{h} \geq 0$, $\text{supp } h \subset D$, and $h(0) = 1$. Then $h(x) \leq h(0) = 1$ for $x \in \mathbb{R}^n$.

Let $g \in L^1(\mathbb{T}^n)$ be a non-negative periodic positive definite function satisfying $\hat{g}_0 = 1$ and $\text{supp } g \subset B_\varepsilon$ for some small positive ε . For example, we can take the periodization of function (13). In this case,

$$0 \leq \hat{g}_\nu \leq \hat{g}_0 = 1, \quad \nu \in \mathbb{Z}^n.$$

Now we set

$$u(x) = \sum_{\nu \in \mathbb{Z}^n} \hat{g}_\nu h(x - \nu), \quad x \in \mathbb{R}^n.$$

Then

$$\hat{u}(\xi) = \hat{h}(\xi) \sum_{\nu \in \mathbb{Z}^n} \hat{g}_\nu e(\nu\xi) = \hat{h}(\xi)g(\xi) \geq 0, \quad \xi \in \mathbb{R}^n.$$

Let us estimate $I := \int_{\mathbb{R}^n} |f|^2 u dx$. First, we have that

$$I = \sum_{\nu \in \mathbb{Z}^n} \hat{g}_\nu \int_{D+\nu} |f(x)|^2 h(x - \nu) dx \leq \sum_{\nu \in \mathbb{Z}^n} \int_{D+\nu} |f|^2 dx = \int_{D+\mathbb{Z}^n} |f|^2 dx.$$

On the other hand, we have that $fh \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, $\widehat{fh} = \hat{f} * \hat{u} \geq 0$, $\hat{f} \geq 0$, and

$$\begin{aligned} I &= \int_{\mathbb{R}^n} \overline{f(x)} \int_{\mathbb{R}^n} (\widehat{fh})(\xi) e(\xi x) d\xi dx = \int_{\mathbb{R}^n} (\widehat{fh})(\xi) \overline{\hat{f}(\xi)} d\xi \\ &\geq \int_{B_r} (\widehat{fh})(\xi) \hat{f}(\xi) d\xi, \end{aligned}$$

where $r > 0$ is sufficiently large. Now we represent the latter integral as follows

$$\int_{B_r} \widehat{f}(\xi) \int_{\mathbb{R}^n} \widehat{f}(\xi - \eta) \widehat{u}(\eta) d\eta d\xi = \int_{\mathbb{R}^n} \widehat{u}(\eta) F_r(\eta) d\eta = \int_{\mathbb{R}^n} \widehat{h}(\eta) g(\eta) F_r(\eta) d\eta,$$

where

$$F_r(\eta) = \int_{B_r} \widehat{f}(\xi) \widehat{f}(\xi - \eta) d\xi.$$

Note that the functions F_r and \widehat{h} are continuous at the origin, and moreover,

$$\int_{B_\varepsilon} g d\xi = \widehat{g}_0 = 1$$

and $g \geq 0$. Therefore, letting $\varepsilon \rightarrow 0$, we get by the second mean-value theorem that

$$I \geq \widehat{h}(0) F_r(0) = \widehat{h}(0) \int_{B_r} (\widehat{f}(\xi))^2 d\xi = \widehat{h}(0) \|f\|_{L^2(B_r)}^2.$$

Let us now let $r \rightarrow \infty$. We have

$$I \geq \widehat{h}(0) \|f\|_{L^2(\mathbb{R}^n)}^2 = \widehat{h}(0) \|f\|_{L^2(\mathbb{R}^n)}^2.$$

Thus,

$$\int_{\mathbb{R}^n} |f|^2 dx \leq (\widehat{h}(0))^{-1} \int_{D+\mathbb{Z}^n} |f|^2 dx.$$

Maximizing $\widehat{h}(0)$, we obtain (27). \square

PROOF OF REMARK 2. It is enough to consider the function

$$f_r(x) = f(x) h_*(r^{-1}x), \quad x \in \mathbb{R}^n, \quad r > 0,$$

where $h_* = |B_1|^{-1} \chi_{B_1} * \chi_{B_1}$, cf. (13). Then $f_r \in L^1(\mathbb{R}^n)$ is a positive definite function with compact support. Therefore, inequality (27) holds for such f_r . Since $h_*(r^{-1}x) \rightarrow h_*(0) = 1$ as $r \rightarrow \infty$ uniformly on any compact subset of \mathbb{R}^n , we arrive at inequality (27) for f . \square

6. Final remarks

1. In $L^p(\mathbb{T}^n)$, Wiener's theorem states that for $f \in L_+^1(\mathbb{T}^n)$ one has

$$\int_{\mathbb{T}^n} |f|^p dx \leq C_{n,p}(D) \int_D |f|^p dx,$$

if and only if p is an even number; see [Wa, Sh] for $n = 1$. Here $D \subset \mathbb{T}^n$ is an origin-symmetric convex body.

Similarly to Wiener's problem for $p = 2$, we state the following question: to find

$$W_{n,p}(D) := \sup_{f \in L_+^1(\mathbb{T}^n) \setminus \{0\}} \frac{\int_{\mathbb{T}^n} |f|^p dx}{|D|^{-1} \int_D |f|^p dx},$$

where p is an even integer.

First, we note that

$$1 \leq W_{n,p}(D) \leq W_{n,2}(D) \leq 2^n, \quad p = 2k, \quad k \in \mathbb{N}.$$

This follows immediately from the fact that f^k is the positive definite function. Thus, to estimate $W_{n,p}(D)$ from above, one can apply any upper estimate of $W_{n,2}(D)$, for example, given by Theorems 2 and 3.

Second, the proof of Theorem 4 can be easily modified to the case $p > 0$. This is because of the fact that crucial relations (23), (24), and (25) can be written as follows:

$$\begin{aligned} f^p(x) &= |\Gamma|^{-p} \sum_{\gamma \in \Gamma} \psi_\varepsilon^p(x - \gamma), \\ \int_{\mathbb{T}^n} f^p dx &= |\Gamma|^{-p+1} \|\psi_\varepsilon\|_{L^p(B_\varepsilon)}^p, \\ \int_D f^p dx &= |\Gamma|^{-p} \|\psi_\varepsilon\|_{L^p(B_\varepsilon)}^p. \end{aligned}$$

This immediately gives

THEOREM 4'. *Let $D \subset \delta I^n$ for $\delta \in (0, 1/q)$, $q = 2, 3, \dots$. Then*

$$|D|q^n \leq W_{n,p}(D).$$

2. In [AT], it was proved that if $L^p(\mathbb{T}) \subset X$ and X is a solid, then

$$L_{\text{loc}+}^p(\mathbb{T}) := \left\{ f \in L_+^1(\mathbb{T}) : \int_{-\delta}^{\delta} |f|^p dx < \infty \right\} \subset X.$$

Recall that a space of functions X is called solid if it satisfies the following property: For every $f = \sum c_\nu e(\nu x)$ in X , if another function $g = \sum d_\nu e(\nu x)$ satisfies $|d_\nu| \leq c_\nu$ for every ν then g is also in X . In particular, if $1 < p \leq 2$, then

$$L_{\text{loc}+}^p(\mathbb{T}^n) \subset l^{p'}(\mathbb{T}^n).$$

Therefore, one seeks the optimal constant

$$\mathcal{W}_{n,p}(D) := \sup_{f \in L_{\text{loc}+}^p(\mathbb{T}^n) \setminus \{0\}} \frac{(\sum_\nu \hat{f}_\nu^{p'})^{1/p'}}{(|D|^{-1} \int_D |f|^p dx)^{1/p}},$$

where $1 < p \leq 2$.

3. N. Wiener in the 1930's [Wi] was the first to study problem (1) for trigonometric series with lacunary coefficients. Later on, this interesting problem received much attention by many authors including A. E. Ingham [In], [Se, Art. 45, Sec. 20, p. 224, (20.38)], J. D. Vaaler [Va], A. Bonami and Sz. Révész [BR] and, A. Babenko and V. Yudin [BY].

4. In the case of $p = \infty$, Wiener's theorem becomes a well-known theorem of Paley [Pa]: if $f \in L_{\text{loc}+}^p$ and f is an even function, then f is continuous on \mathbb{T} and its Fourier series converges uniformly and absolutely. In this case Wiener's problem is closely related to a well-known pointwise Turán problem studied in [ABB, KR3, II].

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